

INJECTIVE STABILIZATION OF ADDITIVE FUNCTORS. II. (CO)TORSION AND THE AUSLANDER-GRUSON-JENSEN FUNCTOR

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ABSTRACT. The formalism of injective (projective) stabilization of additive functors is used to define the notions of the torsion submodule of a module and cotorsion quotient module of a module in complete generality – the new concepts are defined for any module over any ring. General properties of these constructs are established. It is shown that the Auslander-Gruson-Jensen functor applied to the cotorsion functor returns the torsion functor. If the injective envelope of the ring is finitely presented, then the right adjoint of the Auslander-Gruson-Jensen functor applied to the torsion functor returns the cotorsion functor. This correspondence establishes a duality between torsion and cotorsion over such rings. In particular, this duality applies to artin algebras.

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1. INTRODUCTION

This paper is second in a series of three dealing with the injective stabilization of the tensor product and other additive functors. Part 1 [12] provided basic results on this construct, and now we give first applications – definitions of the torsion submodule of a module and of the cotorsion quotient module of a module. The main point here is that these definitions are given in complete generality, without any restrictions on rings or modules. For ease of reference, we sporadically (and somewhat inconsistently) refer to the new torsion as the “injective torsion”; more often though we simply use the term “torsion”.

In Section 2, we define the torsion submodule of a module as the injective stabilization of the tensor product with the module, evaluated at the ring, and establish a dozen or so expected properties of this operation. The new definition is flexible enough to allow for immediate definitions of torsion modules and of torsion-free modules. Over commutative domains the injective torsion coincides with the classical torsion, and for finitely presented modules over arbitrary rings it coincides with

the 1-torsion, defined at each component as the kernel of the canonical map from the module to its bidual. While the 1-torsion is defined over arbitrary rings and, for finitely presented modules over commutative domains, coincides with the classical torsion, this is no longer true for infinite modules. Thus the injective torsion is in general different from the 1-torsion. However, the former is always contained in the latter, and it turns out that the injective torsion functor commutes with filtered colimits. In fact, it is the largest subfunctor of the 1-torsion functor commuting with colimits. As an immediate application, we show that the injective torsion is a radical: the torsion of the module modulo its torsion submodule is zero. If the injective envelope of the ring is flat, then the torsion subfunctor is idempotent, and the torsion-free class is closed under extensions.

In Section 3 we introduce the notions of the cotorsion quotient module of a module and of the cotorsion-free submodule of a module. We also define the cotorsion class and the cotorsion-free class. These new concepts are developed in similarity with our definition of torsion, except that we were not able to find any prior attempts, even over commutative domains, to define the notion of the cotorsion module of a module. There have been attempts to define cotorsion modules and cotorsion-free modules in various degrees of generality, but a quick analysis of the literature reveals considerable confusion in the established sources, even to the point of mixing cotorsion-free modules and cotorsion modules together.

In Section 4, we show that the functor discovered (independently) by Auslander and Gruson-Jensen converts torsion into cotorsion. Under the additional assumption that the injective envelope of the ring is finitely presented, we show that the right adjoint of the Auslander-Gruson-Jensen functor converts cotorsion into torsion, thereby establishing a duality between the two concepts. In particular, this duality holds over artin algebras. For algebras over commutative rings, we utilize the generalized Auslander-Reiten formula, proved in Part 1, to provide additional connections between torsion and cotorsion. In particular, specializing to integer coefficients, we have that the character module of the torsion of a module is isomorphic to the cotorsion of the character module of the module.

The terminology and notation used in this part are the same as in the first paper [12] of the series.

2. TORSION OVER ARBITRARY RINGS

Let Λ be a ring and A a right Λ -module. We return to the injective stabilization $A \otimes \overline{_}$ of $A \otimes _$ and want to take a closer look at its Λ -component. Our goal is to extend the classical torsion, originally defined for modules over commutative domains, to arbitrary modules over arbitrary rings.

If Λ is a commutative domain and K is its field of fractions, then the classical torsion of A coincides with the kernel of the canonical map $A \rightarrow A \otimes K$. But K , being divisible and torsion-free, is also the injective envelope of Λ , and therefore the kernel is isomorphic to the injective stabilization of the functor $A \otimes _$ evaluated at Λ . This simple-minded observation leads to the following definition.

Definition 2.1. *The (injective) torsion of the right Λ -module A is defined by*

$$\mathfrak{s}(A) := (A \otimes \overline{_})(\Lambda) = A \otimes \Lambda.$$

Immediately from this definition we have

Proposition 2.2. *If Λ is a commutative domain, then the injective torsion coincides with the classical torsion.* \square

By [12, Remark 8.5], the injective stabilization of the tensor product is a bifunctor. Together with the canonical isomorphism $A \otimes \Lambda \cong A$ and the fact that, in each component, the inclusion map from the injective stabilization to the module is a module homomorphism [12, Remark 3.3], this yields

Proposition 2.3. *\mathfrak{s} is a subfunctor of the identity functor on the category $\text{Mod-}\Lambda$ of right Λ -modules. In particular, $\mathfrak{s}(A)$ is a submodule of A .* \square

Corollary 2.4. *\mathfrak{s} preserves monomorphisms. In particular, if $\mathfrak{s}(A) = 0$ and B is a submodule of A , then $\mathfrak{s}(B) = 0$.* \square

Definition 2.5. *The submodule $\mathfrak{s}(A)$ will be called the torsion submodule of A .*

Remark 2.6. In general, \mathfrak{s} is not idempotent. It was shown in [11, Proposition 17] that, over a commutative artinian local ring, $\mathfrak{s}^2(A)$ is strictly contained in $\mathfrak{s}(A)$ for any finitely generated nonzero A (here $\mathfrak{s}(A) = \mathfrak{t}(A)$).

There is another possible candidate, $\mathfrak{t}(A)$, for the torsion submodule of A , introduced by Bass. It is defined by the exact sequence

$$0 \longrightarrow \mathfrak{t}(A) \longrightarrow A \xrightarrow{e_A} A^{**}$$

where e_A is the canonical evaluation map. We shall refer to it as the 1-torsion submodule of A .¹ Clearly, \mathfrak{t} is also a subfunctor of the identity functor. The 1-torsion is defined for any module and coincides with the classical torsion, i.e., with the injective torsion, for finitely generated modules over commutative domains. However, this is no longer true for infinitely generated modules. In fact, as the next example shows, the two could be at the extreme ends in terms of size.

Example 2.7. Let $\Lambda := \mathbb{Z}$ and $A := \mathbb{Q}$. Since \mathbb{Q} is divisible, $\mathbb{Q}^{**} = \{0\}$, and therefore $\mathfrak{t}(\mathbb{Q}) = \mathbb{Q}$. On the other hand, \mathbb{Q} is flat and, by [12, Proposition 6.2], $\mathfrak{s}(\mathbb{Q}) = \{0\}$.

While the injective torsion \mathfrak{s} doesn't have this drawback and is indeed our choice for torsion, the 1-torsion functor \mathfrak{t} will also play an important role in our arguments, so now we want to clarify the relationship between the two. The next result puts Example 2.7 in a conceptual framework.

Proposition 2.8. *$\mathfrak{s} \subseteq \mathfrak{t}$, i.e., injective torsion is a subfunctor of 1-torsion.*

Proof. Let A be an arbitrary module. Evaluating the natural transformation

$$A \otimes - \xrightarrow{\mu} (A^*, -)$$

on the injective envelope $\iota : \Lambda \rightarrow I$ of Λ , we have a commutative square

$$\begin{array}{ccc} A \otimes \Lambda & \xrightarrow{1 \otimes \iota} & A \otimes I \\ \mu_\Lambda \downarrow & & \downarrow \mu_I \\ (A^*, \Lambda) & \xrightarrow{(1, \iota)} & (A^*, I) \end{array}$$

¹The case of a vanishing 1-torsion was investigated in [13]; it provides a generalization and a conceptual framework for the notion of linkage of algebraic varieties. That approach was extended to functors by the second author in [16]. The case of a non-vanishing 1-torsion of finitely presented modules over semiperfect rings was studied in [11].

Here $\mathfrak{s}(A) = \text{Ker}(1 \otimes \iota)$ and $\mathfrak{t}(A) = \text{Ker} \mu_\Lambda$. By the left-exactness of the Hom functor, $(1, \iota)$ is monic. The desired result now follows from the commutativity of the square. \square

Returning to finitely presented modules, we have the following important result

Proposition 2.9. *If A is finitely presented, then $\mathfrak{s}(A) = \mathfrak{t}(A)$.*

Proof. This is the isomorphism from the last assertion of [12, Proposition 8.4] evaluated at Λ . \square

Next we want to show that the injective torsion functor commutes with filtered colimits and coproducts. This is a consequence of the following well-known fact.

Lemma 2.10. *Let X, Y be left modules and $-\otimes X \xrightarrow{\alpha} -\otimes Y$ be a natural transformation between the tensor functors on the category $\text{Mod}-\Lambda$ of all right modules. Then the functor $F : \text{Mod}(\Lambda) \rightarrow \text{Ab}$ defined by the exact sequence*

$$0 \longrightarrow F \longrightarrow -\otimes X \xrightarrow{\alpha} -\otimes Y$$

commutes with filtered colimits and coproducts.

Proof. Let $A = \varinjlim A_i$ be a left module represented as the filtered colimit of finitely presented modules. Since both $-\otimes X$ and $-\otimes Y$ commute with coproducts and filtered colimits, we have an exact sequence

$$0 \longrightarrow F(A) \longrightarrow \varinjlim (A_i \otimes X) \xrightarrow{\alpha_A} \varinjlim (A_i \otimes Y)$$

Since the functor \varinjlim is exact in the category of abelian groups, there is also an exact sequence

$$0 \longrightarrow \varinjlim F(A_i) \longrightarrow \varinjlim (A_i \otimes X) \xrightarrow{\alpha_A} \varinjlim (A_i \otimes Y)$$

It follows that

$$\varinjlim F(A_i) \simeq F(A)$$

i.e., F commutes with filtered colimits. An entirely similar argument shows that $F(\coprod M_i) \simeq \coprod F(M_i)$ since taking coproducts is also exact in the category of abelian groups and the tensor product functors commute with coproducts. \square

Corollary 2.11. *For any left module B , the functor*

$$-\overset{\rightharpoonup}{\otimes} B : \text{Mod}(\Lambda) \rightarrow \text{Ab}$$

commutes with filtered colimits and coproducts. In particular, the torsion functor $\mathfrak{s} = -\overset{\rightharpoonup}{\otimes} \Lambda$ commutes with filtered colimits and coproducts.

Proof. Indeed, $-\overset{\rightharpoonup}{\otimes} B$ is determined by the exact sequence of functors

$$0 \rightarrow -\overset{\rightharpoonup}{\otimes} B \rightarrow -\otimes B \rightarrow -\otimes I$$

\square

We can now see what makes \mathfrak{t} different from \mathfrak{s} : the 1-torsion does not, in general, preserve filtered colimits. To show this, we return to Example 2.7. Represent \mathbb{Q} as a filtered colimit of its finitely generated submodules: $\mathbb{Q} = \varinjlim A_i$. Then $\mathfrak{t}(A_i)$ agrees with the classical torsion of A_i , which is zero. If \mathfrak{t} preserved filtered colimits, we would have $\mathfrak{t}(\mathbb{Q}) = \{0\}$, a contradiction.

The next result makes the previous observation precise.

Proposition 2.12. *The torsion functor \mathfrak{s} is the largest subfunctor of the 1-torsion functor \mathfrak{t} that commutes with filtered colimits.*

Proof. We have already seen in Proposition 2.9 that the inclusion $\mathfrak{s} \subseteq \mathfrak{t}$ evaluates to an equality on finitely presented modules. We have also seen in Corollary 2.11 that \mathfrak{s} commutes with filtered colimits. Suppose that $\mu : u \rightarrow t$ is a subfunctor of \mathfrak{t} commuting with filtered colimits. Let $A = \varinjlim A_i$ be a right module expressed as a filtered colimit of finitely presented modules. The components $\mu_i : u(A_i) \rightarrow t(A_i) = \mathfrak{s}(A_i)$ induce a family of monomorphisms $\tilde{\mu}_i : u(A_i) \rightarrow \mathfrak{s}(A_i)$. By applying the exact functor \varinjlim to this family we have a monomorphism $\varinjlim \tilde{\mu}_i : \varinjlim u(A_i) \rightarrow \varinjlim \mathfrak{s}(A_i)$. Since u and \mathfrak{s} commute with filtered colimits, this monomorphism is equivalent to a monomorphism $u(A) \rightarrow \mathfrak{s}(A)$. This is easily seen to be natural in A establishing that $u \subseteq \mathfrak{s}$. \square

One property of torsion shared by both the classical torsion over commutative domains and the 1-torsion over arbitrary rings, is that both functors are radicals, i.e., the quotient of any module modulo its torsion submodule has zero torsion. Now we show that \mathfrak{s} has this property, too.

Theorem 2.13. *\mathfrak{s} is a radical, i.e., $\mathfrak{s}(A/\mathfrak{s}(A)) \simeq \{0\}$ for any module A .*

Proof. As we just mentioned, for any A , $\mathfrak{t}(A/\mathfrak{t}(A)) = \{0\}$, where \mathfrak{t} is the 1-torsion functor. Since \mathfrak{s} is a subfunctor of \mathfrak{t} by Proposition 2.8, we have $\mathfrak{s}(A/\mathfrak{t}(A)) = \{0\}$ for any A . Assuming now that A is finitely presented we have, since $\mathfrak{s}(A) = \mathfrak{t}(A)$ by Proposition 2.9, the desired assertion for finitely presented modules.

If A is now arbitrary, let $A = \varinjlim A_i$ be a representation of A as a filtered colimit of finitely presented modules. Then $\mathfrak{s}(A_i/\mathfrak{s}(A_i)) = \{0\}$ for each A_i and

$$\begin{aligned} \mathfrak{s}(A/\mathfrak{s}(A)) &\simeq \mathfrak{s}\left(\frac{\varinjlim A_i}{\mathfrak{s}(\varinjlim A_i)}\right) \\ &\simeq \mathfrak{s}\left(\frac{\varinjlim A_i}{\varinjlim \mathfrak{s}(A_i)}\right) && \text{(since } \mathfrak{s} \text{ commutes with filtered colimits)} \\ &\simeq \mathfrak{s}\left(\varinjlim \frac{A_i}{\mathfrak{s}(A_i)}\right) && \text{(since colimit is an exact functor)} \\ &\simeq \varinjlim \mathfrak{s}\left(\frac{A_i}{\mathfrak{s}(A_i)}\right) = \{0\} \end{aligned}$$

\square

It is helpful to update our terminology and introduce the familiar classical notions of torsion module and torsion-free module into the new context. As before, let $\iota : {}_\Lambda \Lambda \rightarrow I$ be the injective envelope.

Definition 2.14. *A right module A is said to be a **torsion module** if the canonical monomorphism $\mathfrak{s}(A) \rightarrow A$ is an isomorphism or, equivalently, the map*

$$1 \otimes \iota : A \otimes \Lambda \rightarrow A \otimes I$$

is zero.

Definition 2.15. A module A is said to be *torsion-free* if $\mathfrak{s}(A) = \{0\}$ or, equivalently, the map $1 \otimes \iota : A \otimes \Lambda \rightarrow A \otimes I$ is monic.

The short exact sequence of endofunctors

$$(2.1) \quad 0 \longrightarrow \mathfrak{s} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1}/\mathfrak{s} \longrightarrow 0$$

on the category of all Λ -modules makes it reasonable to set $\mathfrak{s}^{-1} := \mathbf{1}/\mathfrak{s}$. Theorem 2.13 justifies the following definition.

Definition 2.16. The quotient module $\mathfrak{s}^{-1}(A)$ will be called the *torsion-free quotient module* of A .

It is a general property of radical functors that the corresponding torsion-free class² is a reflective subcategory. In our case, this means that \mathfrak{s}^{-1} , called a reflector and viewed as a functor from the category of all modules to the category of torsion-free modules (but not all modules!) is left adjoint to the inclusion functor. Thus, we have

Proposition 2.17. The functor \mathfrak{s}^{-1} from the category of all modules to the category of torsion-free modules preserves all colimits. In particular, the class of torsion-free modules has all colimits. \square

Remark 2.18. It is tempting to use the short exact sequence (2.1) to show that \mathfrak{s} preserves all colimits, too. However such an argument would be incorrect since in that sequence \mathfrak{s}^{-1} denotes a functor from the category of all modules to itself, whereas the codomain of \mathfrak{s}^{-1} in the corollary is the subcategory of torsion-free modules. In fact, as we shall see in Proposition 2.20 below, \mathfrak{s} preserves all colimits if and only if it is the zero functor. It is a general property of radicals that colimits in the reflective subcategory can be computed as images under the reflector (in our case, \mathfrak{s}^{-1}) of colimits in the ambient category.

The next three results deal with the vanishing and the exactness properties of the torsion functor. Since tensoring with a flat module is an exact functor, we have

Lemma 2.19. If A is flat, then $\mathfrak{s}(A) = \{0\}$, i.e., each flat module is torsion-free. \square

Proposition 2.20. The following conditions are equivalent:

- a) \mathfrak{s} preserves epimorphisms;
- b) \mathfrak{s} is the zero functor;
- c) ${}_{\Lambda}\Lambda$ is absolutely pure;
- d) Λ is *left FP-injective*, i.e., $\text{Ext}_{\Lambda}^1(M, \Lambda) = \{0\}$ for all finitely presented left Λ -modules M .

In particular, if Λ is selfinjective on the left, then \mathfrak{s} is the zero functor.

Proof. Given any A , choose an epimorphism $P \rightarrow A \rightarrow 0$ with P projective. Assuming that \mathfrak{s} preserves epimorphisms, we have an epimorphism $\mathfrak{s}(P) \rightarrow \mathfrak{s}(A) \rightarrow 0$. By Lemma 2.19, $\mathfrak{s}(P) = \{0\}$ and, therefore, $\mathfrak{s}(A) = \{0\}$. This proves the implication a) \Rightarrow b). The converse is trivial. To prove the remaining equivalences, notice that \mathfrak{s} is the zero functor if and only if $A \otimes \Lambda \rightarrow A \otimes I$ is monic for any A . In terminology of [10], this means that ${}_{\Lambda}\Lambda$ is absolutely pure, which is equivalent to the condition $\text{Ext}_{\Lambda}^1(M, \Lambda) = \{0\}$ for all finitely presented left Λ -modules M [15, Proposition 1]. \square

²Strictly speaking, this should be called the **pretorsion-free** class.

Proposition 2.21. *If $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is a pure exact sequence, then the sequence*

$$0 \rightarrow \mathfrak{s}(A') \rightarrow \mathfrak{s}(A) \rightarrow \mathfrak{s}(A'') \rightarrow 0$$

is exact.

Proof. Follows immediately from the snake lemma. \square

Now we want to discuss the relationship between the torsion functor \mathfrak{s} and flat modules.

Proposition 2.22. *Suppose the injective envelope of ${}_{\Lambda}\Lambda$ is flat. Then*

- (1) $\mathfrak{s} \simeq \text{Tor}_1(_, \Sigma\Lambda)$.
- (2) *If $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is a short exact sequence of right Λ -modules, then the induced sequence*

$$0 \rightarrow \mathfrak{s}(A') \rightarrow \mathfrak{s}(A) \rightarrow \mathfrak{s}(A'') \rightarrow A' \otimes \Sigma\Lambda \rightarrow A \otimes \Sigma\Lambda \rightarrow A'' \otimes \Sigma\Lambda \rightarrow 0$$

is exact. In particular, \mathfrak{s} is left-exact.
- (3) $\mathfrak{s}^2 = \mathfrak{s}$, i.e., \mathfrak{s} is the torsion class of a torsion theory.
- (4) *The torsion-free class is closed under extensions.*

Proof. (1) Tensor the short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow I \longrightarrow \Sigma\Lambda \longrightarrow 0$$

with an arbitrary right module and pass to the long exact sequence.

(2) Follows from the snake lemma.

(3) By (2), \mathfrak{s} is left-exact. Thus the short exact sequence

$$0 \rightarrow \mathfrak{s}(A) \rightarrow A \rightarrow \mathfrak{s}^{-1}(A) \rightarrow 0$$

gives rise to the exact sequence

$$0 \rightarrow \mathfrak{s}^2(A) \rightarrow \mathfrak{s}(A) \rightarrow \mathfrak{s}(\mathfrak{s}^{-1}(A))$$

where $\mathfrak{s}(\mathfrak{s}^{-1}(A)) = \{0\}$ by Theorem 2.13.

(4) Apply \mathfrak{s} to a short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$. The resulting exact sequence $0 \rightarrow \mathfrak{s}(A') \rightarrow \mathfrak{s}(A) \rightarrow \mathfrak{s}(A'')$ shows that if $\mathfrak{s}(A')$ and $\mathfrak{s}(A'')$ vanish, then so does $\mathfrak{s}(A)$, which means that the torsion-free class is closed under extensions. \square

Now we want to give another description of the torsion submodule, this time using the notion of colimit extension. Let \mathcal{F} denote the class of flat (right) Λ -modules, and $\text{Rej}(A, \mathcal{F})$ – the reject of \mathcal{F} in the right module A . The restriction of $\text{Rej}(A, \mathcal{F})$ to the (full) subcategory $\text{mod}(\Lambda)$ determined by the finitely presented modules will be denoted by $\text{rej}(A, \mathcal{F})$.

Proposition 2.23. $\mathfrak{s} \simeq \overrightarrow{\text{rej}}(_, \mathcal{F})$, i.e., the torsion functor is isomorphic to the colimit extension of the reject of flats restricted to finitely presented modules.

Proof. Recall that the 1-torsion functor \mathfrak{t} can be defined as the reject of Λ . As Λ is flat, $\text{Rej}(_, \mathcal{F})$ is a subfunctor of \mathfrak{t} . On the other hand, let F be a flat module and

$f : A \rightarrow F$ an arbitrary homomorphism. Since torsion is a subfunctor, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{s}(A) & \longrightarrow & A \\ \downarrow \mathfrak{s}(f) & & \downarrow f \\ \mathfrak{s}(F) & \longrightarrow & F \end{array}$$

By Lemma 2.19, $\mathfrak{s}(F) = \{0\}$, and therefore torsion is a subfunctor of the reject. Thus we have a chain of functors

$$\mathfrak{s} \subseteq \text{Rej}(_, \mathcal{F}) \subseteq \mathfrak{t}$$

Restricting it to finitely presented modules we have an identification of the end terms. Therefore $\text{rej}(_, \mathcal{F})$ coincides with \mathfrak{s} restricted to finitely presented modules. Passing to the colimit extensions and using the fact that \mathfrak{s} preserves filtered colimits, we have the desired result. \square

This alternative description of the torsion functor \mathfrak{s} , together with the notion of right cosatellite and [12, Proposition 4.6], shows that $\mathfrak{s}^{-1}(A) \simeq C^1(A \otimes _)(\Lambda)$, which allows us to rewrite the short exact sequence

$$0 \rightarrow \mathfrak{s}(A) \rightarrow A \rightarrow \mathfrak{s}^{-1}(A) \rightarrow 0$$

in the following form

$$(2.2) \quad 0 \longrightarrow \overrightarrow{\text{rej}}(A, \mathcal{F}) \longrightarrow A \longrightarrow C^1(A \otimes _)(\Lambda) \longrightarrow 0.$$

In summary,

$$(2.3) \quad \mathfrak{s}(A) = \overrightarrow{\text{rej}}(A, \mathcal{F}) \quad \text{and} \quad \mathfrak{s}^{-1}(A) = C^1(A \otimes _)(\Lambda)$$

Corollary 2.24. *The right cosatellite $C^1(A \otimes _)(\Lambda)$, viewed as a functor of A from all modules to torsion-free modules, preserves all colimits.*

Proof. Follows from Proposition 2.17. \square

We end the discussion of torsion by looking at its iteration. This yields a descending chain

$$(2.4) \quad A \supset \mathfrak{s}(A) \supset \dots \supset \mathfrak{s}^n(A) \supset \dots$$

of submodules of A . By [12, Lemma 3.2], these submodules are determined uniquely not only as modules up to isomorphism but also as submodules. It would be of interest to investigate this chain and, in particular, its asymptotic properties. We only know its behavior in some simple cases. Recapping Lemma 2.19, Proposition 2.20, Proposition 2.22, and Corollary 2.4, we have:

- If A is flat, then $\mathfrak{s}(A) = \{0\}$.
- If Λ is selfinjective, then $\mathfrak{s}(A) = \{0\}$ for all A .
- If the injective envelope of Λ is flat, then $\mathfrak{s}^2 = \mathfrak{s}$ and the chain stabilizes at the first step. This generalizes the same property for classical torsion over commutative domains.
- The injective torsion of any submodule of a flat module is zero. In particular, this holds for syzygy modules in projective resolutions.

It is easy to construct modules A with $\mathfrak{s}(A) = A$. This happens, for example, when $A \otimes I = \{0\}$, where I is the injective envelope of Λ . Over an arbitrary ring, all nonzero finitely presented modules which coincide with their injective torsion (i.e., with their 1-torsion) can be characterized: each such module is the transpose of a module of projective dimension one [11, Proposition 5].

Finally, assume that Λ is a commutative artinian local ring and A an arbitrary finitely generated Λ -module. It was shown in [11, Proposition 17] that $\mathfrak{t}(A)$ does not contain minimal generators of A . Since in this case $\mathfrak{s}(A) = \mathfrak{t}(A)$, we have that: a) $\mathfrak{s}(A)$ is a proper submodule of A , b) the chain (2.4) stabilizes at $\{0\}$, and c) the length of the chain does not exceed the Loewy length of A .

3. COTORSION OVER ARBITRARY RINGS

Given the utmost generality of our definition of torsion, one may ask if that definition can be formally dualized to yield a definition of cotorsion. The answer is yes and, as we shall see, this can be done with little effort. However, compared with torsion, one faces the surprising fact that while there are several competing definitions of cotorsion modules, there seems to be no definition of the cotorsion module of a module. Our next goal is to provide such a definition and do it in the utmost generality - for any module over any ring.

Recall that the injective torsion submodule $\mathfrak{s}(A) = (A \otimes \overline{})(\Lambda)$ of a right Λ -module A was defined as the kernel of the map $A \otimes \Lambda \rightarrow A \otimes I$, where I is an injective container of ${}_{\Lambda}\Lambda$. The fact that $\mathfrak{s}(A)$ is a submodule of A is due to the canonical isomorphism $A \otimes \Lambda \cong A$. In a dual approach, it could be expected that one should use the other well-known canonical isomorphism, $\text{Hom}(\Lambda, C) \cong C$, where C is an arbitrary (say, left) Λ -module. This leads to the contravariant functor $\text{Hom}(_, C)$. Since torsion was defined as the **injective** stabilization of the functor $A \otimes _$, in a dual approach one should be looking at the **projective** stabilization of $\text{Hom}(_, C)$. Viewing $\text{Hom}(_, C)$ as a **covariant** functor on the opposite category (which is never a module category but is still abelian) one is led to consider the contravariant Hom functor modulo injectives, customarily denoted by $\overline{\text{Hom}}$. This motivates

Definition 3.1. *Let C be a (left) Λ -module. The cotorsion module of C is defined as $\mathfrak{q}(C) := \overline{\text{Hom}}(_, C)(\Lambda) = \overline{\text{Hom}}(\Lambda, C)$.³*

In more detail, if

$$(3.1) \quad 0 \longrightarrow \Lambda \xrightarrow{\iota} I \longrightarrow \Sigma\Lambda \longrightarrow 0$$

is an injective cosyzygy sequence, then the cotorsion module of C is defined by the exact sequence

$$(3.2) \quad 0 \longrightarrow (\Sigma\Lambda, C) \longrightarrow (I, C) \xrightarrow{(\iota, C)} (\Lambda, C) \longrightarrow (\overline{\Lambda}, C) \longrightarrow 0,$$

where we have switched to the categorical notation.

The reader should verify that the contravariant Hom modulo injectives fits the pattern dual to that of the injective stabilization. In fact, for any additive functor F from modules to abelian groups, the projective stabilization of F is defined as the

³The term in the middle denotes the projective stabilization of the **contravariant** functor $\text{Hom}(_, C)$.

cokernel of the natural transformation $L_0 F \rightarrow F$. If F is a contravariant Hom functor, then each component of its projective stabilization is precisely the classes of homomorphisms modulo the ones factoring through injectives. As a consequence, we have

Proposition 3.2. *q is a quotient functor of the identity functor on the category $\text{Mod}(\Lambda)$. In particular, $q(C)$ is a quotient module of C .* \square

Corollary 3.3. *q preserves epimorphisms. In particular, if $q(D) = \{0\}$ and C is a quotient module of D , then $q(C) = \{0\}$.* \square

Definition 3.4. *The quotient module $q(C)$ will be called the cotorsion quotient module of C .*

Remark 3.5. The reader should not think that $q^2 = q$. In plain terms, there may be nonzero maps $\Lambda \rightarrow q(C)$ factoring through injectives. A counterexample, based on a duality argument, will be provided by Remark 4.14.

The defining sequence (3.2) gives rise to a short exact sequence of endofunctors on the category of Λ -modules

$$0 \longrightarrow q^{-1} \longrightarrow \mathbf{1} \longrightarrow q \longrightarrow 0$$

which we use to define the endofunctor q^{-1} . We can rewrite it as a short exact sequence

$$0 \longrightarrow I(\Lambda, C) \longrightarrow (\Lambda, C) \longrightarrow (\overline{\Lambda}, \overline{C}) \longrightarrow 0$$

of Λ -modules, where $I(\Lambda, C)$ denotes the submodule of (Λ, C) consisting of the maps factoring through injectives.

Lemma 3.6. *Under the canonical isomorphism $(\Lambda, C) \cong C : f \mapsto f(1)$, $I(\Lambda, C)$ identifies with $\text{Tr}(\mathcal{J}, C)$, the trace in C of the class \mathcal{J} of injective Λ -modules.*

Proof. Any map $\Lambda \rightarrow C$ is uniquely determined by the image of the identity. Thus, viewing $I(\Lambda, C)$ as a submodule of C , we have that for any $k \in I(\Lambda, C)$, there is a map $f : \Lambda \rightarrow C$ which factors through an injective and such that $f(1) = k$. Therefore $I(\Lambda, C)$ is contained in the trace. To show the reverse inclusion, suppose that $l \in C$ can be written as $h(m)$ for some map $h : J \rightarrow C$, where J is injective and $m \in J$. Define a map $g : \Lambda \rightarrow J$ by setting $g(1) := m$. Then $l = hg(1)$ and therefore $l \in I(\Lambda, C)$. Since $I(\Lambda, C)$ is a Λ -module, the trace is contained in $I(\Lambda, C)$. \square

Proposition 3.7. *$\text{proj dim Tr}(\mathcal{J}, _) \leq 1$. Moreover, $\text{proj dim Tr}(\mathcal{J}, _) = 0$ if and only if Λ is left selfinjective.*

Proof. The exact sequences (3.2) yields a length one projective resolution

$$0 \longrightarrow (\Sigma\Lambda, _) \longrightarrow (I, _) \longrightarrow \text{Tr}(\mathcal{J}, _) \longrightarrow 0$$

of the trace, proving the first assertion. This sequence and Yoneda's lemma, show that the trace is projective if and only if $\Sigma\Lambda$ is a direct summand of I , which proves the second claim. \square

We shall say that a quotient functor of the identity functor is a **coradical** if it is a radical on the opposite category. In line with the formal duality between torsion and cotorsion, we may ask if there is an analog of Proposition 2.13. The answer is yes and is given by the next result.

Proposition 3.8. \mathfrak{q} is a coradical, i.e., $\mathfrak{q}(\mathfrak{q}^{-1}(C)) = \{0\}$ for any C .

Proof. $\mathfrak{q}(I(\Lambda, C)) = \{0\}$ if and only if any map $g : \Lambda \rightarrow I(\Lambda, C)$ factors through an injective. Let $i : I(\Lambda, C) \rightarrow C$ be the inclusion map. Then $ig(1) = g(1) \in I(\Lambda, C)$, where we identify $I(\Lambda, C)$ with its image in C . Therefore, there is a map $f : \Lambda \rightarrow C$ which factors through an injective and such that $f(1) = ig(1)$. Then $ig = f$ factors through some injective J and we have a commutative diagram

$$\begin{array}{ccccc} \Lambda & \xrightarrow{g} & I(\Lambda, C) & \xrightarrow{i} & C \\ & \searrow & \uparrow & \nearrow & \\ & & J & & \end{array}$$

of solid arrows. By Lemma 3.6, the northeastern map factors through i , making the right-hand side triangle commute. Since i is monic, the left-hand side triangle commutes, too, showing that g factors through an injective. \square

Having a definition of the cotorsion module of a module, we can now define a notion of cotorsion module.

Definition 3.9. A module C is said to be a **cotorsion module** if the canonical surjection $(\Lambda, C) \rightarrow (\overline{\Lambda}, \overline{C})$ is an isomorphism. Equivalently, no nonzero map $\Lambda \rightarrow C$ factors through an injective, or, equivalently, the map (ι, C) in (3.2) is the zero map.

Remark 3.10. As we mentioned before, while we could not find any prior definition of the cotorsion module of a module, there have been attempts to define cotorsion modules in various degrees of generality. In the case Λ is a commutative domain, Matlis [14] calls C a cotorsion module if $\text{Hom}(Q, C) = \{0\} = \text{Ext}^1(Q, C)$, where Q is the quotient field of Λ . In this case, $Q = I$ is the injective envelope of Λ and therefore the map (ι, C) from (3.2) is zero. Thus a cotorsion module in the sense of Matlis is a cotorsion module in the sense of Definition 3.9. In fact, in this case, $C \simeq \mathfrak{q}(C) \simeq \text{Ext}^1(\Sigma\Lambda, C)$, as can be seen by applying $(_, C)$ to the sequence (3.1).

Having defined cotorsion modules, we proceed to define cotorsion-free modules.

Definition 3.11. The module C is said to be **cotorsion-free** if the cotorsion module of C is zero:

$$\mathfrak{q}(C) = (\overline{\Lambda}, \overline{C}) = \{0\}$$

Equivalently, any map $\Lambda \rightarrow C$ factors through an injective, or, equivalently, the map (ι, C) in (3.2) is epic.

Clearly, any injective module is cotorsion-free. It is also clear that if a module is both cotorsion and cotorsion-free, then it is the zero module.

It is a general property of coradical functors that the corresponding cotorsion-free class⁴ is a coreflective subcategory. In our case, this means that \mathfrak{q}^{-1} , called a coreflector and viewed as a functor from the category of all modules to the category of cotorsion-free modules (but not all modules!) is right adjoint to the inclusion functor. Thus, we have

⁴Strictly speaking, this should be called the **precotorsion-free** class.

Proposition 3.12. *The functor \mathfrak{q}^{-1} from the category of all modules to the category of cotorsion-free modules preserves all limits. In particular, the class of cotorsion-free modules has all limits.* \square

Lemma 3.6 and Proposition 3.8 justify the following

Definition 3.13. *The trace in C of the class of injective Λ -modules will be called the cotorsion-free submodule of C .*

Thus a module is cotorsion-free if and only if it coincides with its cotorsion-free submodule.

Summarizing the foregoing discussion, we rewrite the defining short exact sequence for the cotorsion quotient module of a module C using the trace of the class \mathcal{I} of injectives in C :

$$(3.3) \quad 0 \longrightarrow \text{Tr}(\mathcal{I}, C) \longrightarrow C \longrightarrow \mathfrak{q}(C) \longrightarrow 0$$

Recalling the definition of the left cosatellite of a contravariant functor [12, Section 4] and the fact that the Hom functor is left-exact (in fact, we only need the half-exactness), we observe that $\text{Tr}(\mathcal{I}, C)$ is nothing but the left cosatellite of the functor $\text{Hom}(-, C)$ evaluated on Λ , i.e.,

$$\text{Tr}(\mathcal{I}, C) \simeq C_1(-, C)(\Lambda)$$

The same is expressed, by the short exact sequence

$$(3.4) \quad 0 \longrightarrow C_1(-, C)(\Lambda) \longrightarrow C \longrightarrow C/\text{Tr}(\mathcal{I}, C) \longrightarrow 0$$

which is similar to the equation (2.2). In multiplicative notation, the similarity with (2.3) becomes even more apparent:

$$(3.5) \quad \mathfrak{q}(C) = (\text{Tr}(\mathcal{I}, C))^{-1} \quad \text{and} \quad \mathfrak{q}^{-1}(C) = C_1(-, C)(\Lambda).$$

Proposition 3.14. *The functor $\text{Tr}(\mathcal{I}, C) \simeq C_1(-, C)(\Lambda)$, viewed as a functor of C from all modules to cotorsion-free modules, preserves all limits.*

Proof. Follows from Proposition 3.12. \square

Remark 3.15. Enochs and Jenda [5, Definition 5.3.22] call a module M over an arbitrary ring cotorsion if $\text{Ext}^1(F, M) = \{0\}$ for any flat module F . They remark that their definition generalizes the definitions of Harrison [9] and Warfield [17] and agrees with that of Fuchs [6] but differs from the definition of Matlis mentioned above. The last observation is easy to explain: injective modules are cotorsion in the sense of Enochs and Jenda, but in general such modules are not cotorsion in the sense of Matlis. In fact, as our definition shows, a better term for the cotorsion modules in the sense of Enochs-Jenda might have been “cotorsion-free modules” but, even under the new name, among such modules there would be modules which are not cotorsion-free in the sense of Definition 3.11. For example, Enochs and Jenda [5, Lemma 5.3.23] show that every pure injective is cotorsion in their sense. We shall now show that there are pure injectives which are not cotorsion-free in the sense of Definition 3.11.⁵ Let \mathbb{k} be a field, $\Lambda := \mathbb{k}[[X]]$, and $C := \Lambda$. Then [18, Proposition 1] $\mathbb{k}[[X]]$ is pure injective as a module over itself. As $\mathbb{k}[[X]]$ is a PID, any injective is divisible and therefore the trace of the injectives in $\mathbb{k}[[X]]$ consists

⁵The authors are grateful to Gena Puninski for a helpful comment leading to this example.

of divisible elements. Since $\mathbb{k}[[X]]$ is obviously not divisible, the injective trace is properly contained in $\mathbb{k}[[X]]$, and therefore C is not cotorsion-free. In fact, since 0 is the only divisible element in $\mathbb{k}[[X]]$, the latter is cotorsion. Thus the class of cotorsion $\mathbb{k}[[X]]$ -modules in the sense of Enochs-Jenda contains both cotorsion-free modules (e.g., injectives) and cotorsion modules (e.g., $\mathbb{k}[[X]]$).

The next three results deal with the vanishing and the exactness properties of the cotorsion functor. Since mapping into an injective is an exact functor, we have

Lemma 3.16. *If C is an injective left Λ -module, then $\mathfrak{q}(C) = \{0\}$, i.e., each injective is cotorsion-free.* \square

Proposition 3.17. *The following conditions are equivalent:*

- a) \mathfrak{q} preserves monomorphisms;
- b) \mathfrak{q} is the zero functor;
- c) Λ is selfinjective on the left.

Proof. Given any left Λ -module C choose a monomorphism $0 \rightarrow C \rightarrow J$ with J injective. Assuming \mathfrak{q} preserves monomorphisms, we have a monomorphism $0 \rightarrow \mathfrak{q}(C) \rightarrow \mathfrak{q}(J)$. By the previous lemma, $\mathfrak{q}(J) = \{0\}$ and therefore $\mathfrak{q}(C) = \{0\}$. This proves the implication a) \Rightarrow b). The converse is trivial. The implication c) \Rightarrow b) is also trivial. Now assume that $\mathfrak{q} = 0$. Then $\{0\} = \mathfrak{q}(\Lambda)$, which shows that the identity map on Λ factors through injectives. Therefore Λ is injective. \square

Proposition 3.18. *If the sequence $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ is copure exact (i.e., remains exact under an arbitrary covariant Hom functor), then the sequence*

$$0 \rightarrow \mathfrak{q}(C') \rightarrow \mathfrak{q}(C) \rightarrow \mathfrak{q}(C'') \rightarrow 0$$

is exact.

Proof. Follows immediately from the snake lemma. \square

Now we want to discuss the relationship between the cotorsion functor and projective modules.

Proposition 3.19. *Suppose the injective envelope of ${}_{\Lambda}\Lambda$ is projective. Then*

- (1) $\mathfrak{q} \simeq \text{Ext}^1(\Sigma\Lambda, _)$.
- (2) *If $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ is an exact sequence, then the induced sequence*

$$0 \rightarrow (\Sigma\Lambda, C') \rightarrow (\Sigma\Lambda, C) \rightarrow (\Sigma\Lambda, C'') \rightarrow \mathfrak{q}(C') \rightarrow \mathfrak{q}(C) \rightarrow \mathfrak{q}(C'') \rightarrow 0$$

is exact. In particular, \mathfrak{q} is right-exact.
- (3) *The natural transformation $\mathfrak{q}(\mathbf{1} \rightarrow \mathfrak{q}) : \mathfrak{q} \rightarrow \mathfrak{q}^2$ is an isomorphism.*
- (4) *The cotorsion-free class is closed under extensions.*

Proof. (1) Map the short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow I \longrightarrow \Sigma\Lambda \longrightarrow 0$$

into an arbitrary module and pass to the long exact sequence.

(2) Follows from the snake lemma.

(3) By (2), \mathfrak{q} is right-exact. Thus the short exact sequence

$$0 \longrightarrow I(\Lambda, C) \longrightarrow C \longrightarrow \mathfrak{q}(C) \longrightarrow 0$$

gives rise to the exact sequence

$$\mathfrak{q}(I(\Lambda, C)) \longrightarrow \mathfrak{q}(C) \longrightarrow \mathfrak{q}^2(C) \longrightarrow 0.$$

By Proposition 3.8, $\mathfrak{q}(I(\Lambda, C)) \simeq \{0\}$, whence the desired claim.

(4) Apply \mathfrak{q} to a short exact sequence $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$. The resulting exact sequence $\mathfrak{q}(C') \rightarrow \mathfrak{q}(C) \rightarrow \mathfrak{q}(C'') \rightarrow 0$ shows that if the end terms vanish, then so does the middle term. \square

4. TORSION, COTORSION, AND THE AUSLANDER-GRUSON-JENSEN FUNCTOR

Recall that a covariant functor $F : \text{Mod}(\Lambda^{op}) \rightarrow \text{Ab}$ is finitely presented if there are modules X, Y and a sequence of natural transformations

$$(Y, _) \rightarrow (X, _) \rightarrow F \rightarrow 0$$

such that for any left module M , the sequence of abelian groups

$$(Y, M) \rightarrow (X, M) \rightarrow F(M) \rightarrow 0$$

is exact. Given two finitely presented functors F, G , it is easily verified that the natural transformations between F and G form an abelian group.

Theorem 4.1 ([1], Theorem 2.3). *Let $\text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab})$ be the category of all finitely presented functors together with natural transformations between them. Then:*

- (1) *$\text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab})$ is abelian. A sequence of finitely presented functors is exact if and only if it is exact at each component.*
- (2) *The projectives are precisely the representable functors $(M, _)$.*
- (3) *Every finitely presented functor $F \in \text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab})$ has a projective resolution of the form*

$$0 \rightarrow (Z, _) \rightarrow (Y, _) \rightarrow (X, _) \rightarrow F \rightarrow 0$$

\square

Moreover, since $\text{Mod}(\Lambda^{op})$ has enough projectives, $\text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab})$ has enough injectives. This result is due to Ron Gentle and appears in [7], where the existence of injectives is shown in Proposition 1.4 and the discussion following that proposition. The fact that $\text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab})$ has enough injectives allows one to compute derived functors.

Both functor categories $\text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab})$ and $(\text{mod}(\Lambda), \text{Ab})$ contain certain functors which are fundamental to the study of model theory of modules. In order to understand these functors we recall the notion of a finitely presented functor $F : \mathcal{A} \rightarrow \text{Ab}$ for an additive category \mathcal{A} . Such a functor is finitely presented if there are $A, B \in \mathcal{A}$ and an exact sequence

$$(B, _) \rightarrow (A, _) \rightarrow F \rightarrow 0$$

where exactness is understood componentwise, meaning that for any object $C \in \mathcal{A}$, the sequence of abelian groups

$$(B, C) \rightarrow (A, C) \rightarrow F(C) \rightarrow 0$$

is exact. Thus the category $\text{fp}(\text{mod}(\Lambda), \text{Ab})$ consists of all functors $F : \text{mod}(\Lambda) \rightarrow \text{Ab}$ for which there are finitely presented modules X, Y and a presentation

$$(Y, _) \rightarrow (X, _) \rightarrow F \rightarrow 0$$

If the module X is finitely presented, then the representable functor

$$(X, _) : \text{Mod}(\Lambda) \rightarrow \text{Ab}$$

commutes with filtered colimits and is thus the colimit extension of its restriction to finitely presented modules. As a result, given any finitely presented functor $F \in \text{fp}(\text{mod}(\Lambda), \text{Ab})$ with presentation

$$(Y, _) \rightarrow (X, _) \rightarrow F \rightarrow 0$$

the functor $\vec{F} : \text{Mod}(\Lambda) \rightarrow \text{Ab}$ has a presentation

$$(Y, _) \rightarrow (X, _) \rightarrow \vec{F} \rightarrow 0.$$

Hence \vec{F} is finitely presented as a functor on the large module category $\text{Mod}(\Lambda)$. Thus $\text{fp}(\text{mod}(\Lambda), \text{Ab})$ is a subcategory of both the functor category $(\text{mod}(\Lambda), \text{Ab})$ and the functor category $\text{fp}(\text{Mod}(\Lambda), \text{Ab})$, via $F \mapsto \vec{F}$.

Since these functors sit inside the three different functor categories $\text{fp}(\text{mod}(\Lambda), \text{Ab})$, $(\text{mod}(\Lambda), \text{Ab})$, and $\text{fp}(\text{Mod}(\Lambda), \text{Ab})$, where two of these categories consist of finitely presented functors with different domain categories and the other has both finitely presented and non-finitely presented functors, the term finitely presented may become confusing. A first step to escape this quandary is to use the equivalence between $(\text{mod}(\Lambda), \text{Ab})$ and the category of functors on $\text{Mod}(\Lambda)$ that commute with filtered colimits. This allows us to view both $\text{fp}(\text{Mod}(\Lambda), \text{Ab})$ and $(\text{mod}(\Lambda), \text{Ab})$ as consisting of functors on $\text{Mod}(\Lambda)$.

Once this convention is taken, we can identify the functors $F \in \text{fp}(\text{mod}(\Lambda), \text{Ab})$ as being in the intersection of the categories $\text{fp}(\text{Mod}(\Lambda), \text{Ab})$ and $(\text{mod}(\Lambda), \text{Ab})$. We will use the following terminology to discuss these functors. A functor $F : \text{Mod}(\Lambda) \rightarrow \text{Ab}$ is called a **pp-functor** if there exists finitely presented modules X, Y and a presentation

$$(Y, _) \rightarrow (X, _) \rightarrow F \rightarrow 0$$

With this terminology set, the full subcategory of $\text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab})$ consisting of the pp-functors is equivalent to the functor category $\text{fp}(\text{mod}(\Lambda^{op}), \text{Ab})$ and the full subcategory of $(\text{mod}(\Lambda), \text{Ab})$ consisting of all pp-functors is equivalent to the functor category $\text{fp}(\text{mod}(\Lambda), \text{Ab})$. In addition, these two full subcategories are abelian and their inclusions are exact.

The Auslander-Gruson-Jensen duality, discovered by Auslander in [2] and independently by Gruson and Jensen in [8], is a pair of exact contravariant functors

$$\begin{array}{ccc} & D & \\ & \curvearrowright & \\ \text{fp}(\text{mod}(\Lambda^{op}), \text{Ab}) & & \text{fp}(\text{mod}(\Lambda), \text{Ab}) \\ & \curvearrowleft & \\ & D & \end{array}$$

satisfying the following properties:

- (1) If X is a finitely presented left module then

$$D(X, _) \simeq _ \otimes X \quad \text{and} \quad D(_ \otimes X) \simeq (X, _)$$

(2) If X is a finitely presented right module then

$$D(X, _)\simeq X \otimes _ \quad \text{and} \quad D(X \otimes _)\simeq (X, _)$$

There is a functor

$$D_A : \text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab}) \rightarrow (\text{mod}(\Lambda), \text{Ab})$$

defined by

$$D_A := R_0(\epsilon \circ w),$$

where ϵ is the tensor embedding [12, (9.2)]

$$\epsilon : \text{Mod}(\Lambda^{op}) \rightarrow (\text{mod}(\Lambda), \text{Ab}) : M \mapsto _ \otimes M$$

and w is the defect functor [12, (5.1)]. The functor D_A is contravariant, exact, and for any representable functor $(M, _)$

$$D_A(M, _) = _ \otimes M$$

As shown in [4, Proposition 9], the functor D_A is completely determined by these properties.

Theorem 4.2 ([4], Theorems 23 and 29). *The functor*

$$D_A : \text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab}) \rightarrow (\text{mod}(\Lambda), \text{Ab})$$

admits a left adjoint D_L and a right adjoint D_R , both of which are fully faithful. The functors D_R and D_A restrict to the Auslander-Gruason-Jensen duality D on the full subcategories of pp-functors. \square

The foregoing statement is part of the following diagram of functors

$$\begin{array}{ccc}
 & & D_L \\
 & \swarrow & \searrow \\
 \text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab}) & \xrightarrow{D_A} & (\text{mod}(\Lambda), \text{Ab}) \\
 & \nwarrow & \nearrow \\
 & & D_R \\
 & \swarrow & \searrow \\
 & & \text{Mod}(\Lambda^{op})
 \end{array}$$

$\swarrow \quad \searrow \quad \swarrow \quad \searrow$
 $w \quad L^0 \Upsilon \quad R_0 \epsilon \quad \text{ev}_\Lambda \quad \epsilon$
 Υ

Before proceeding, we need to address an issue with notation. Recall that for any functor F , the projective stabilization is denoted by \underline{F} and the injective stabilization is denoted by \overline{F} . One of the interesting results from [3] is that the projective stabilization of the representable $\text{Hom}(B, _)$ yields the functor $\underline{\text{Hom}}(B, _)$ which sends any module C to the abelian group $\underline{\text{Hom}}(B, C)$, known as Hom modulo projectives. In other words, if $F = \text{Hom}(B, _)$, then we have an isomorphism

$$\underline{F} \simeq \underline{\text{Hom}}(B, _)$$

Continuing with the assumption that $F = \text{Hom}(B, _)$, the injective stabilization \overline{F} is 0 because in this case F is left exact. Therefore the injective stabilization of $\text{Hom}(B, _)$ does not return the functor $\overline{\text{Hom}}(B, _)$, which sends any module C to the abelian group $\overline{\text{Hom}}(B, C)$, known as Hom modulo injectives.

Remark 4.3. Though $\overline{\text{Hom}}(B, _)$ is not the injective stabilization of $\text{Hom}(B, _)$, it is injectively stable as it clearly vanishes on injectives. This functor is also finitely presented: just apply the contravariant Hom functor to a cosyzygy sequence of B .

One of the major reasons for looking at the injective stabilization of the tensor product is that this notion is formally dual to that of the projective stabilization of the Hom functor. The projective stabilization $\underline{\text{Hom}}(A, _)$ of the covariant Hom functor is not in general finitely presented, but as we just remarked, the functor $\overline{\text{Hom}}(B, _)$ is finitely presented. The functor D_A sends representable functors to tensor product functors and, as we shall now show, we can recover $_ \otimes B$ using D_A .

Theorem 4.4. *For any module B*

$$D_A \overline{\text{Hom}}(B, _) \simeq _ \otimes B$$

Proof. As seen above, the cosyzygy sequence $0 \rightarrow B \xrightarrow{j} I \rightarrow \Sigma B \rightarrow 0$ yields a presentation

$$(I, _) \rightarrow (B, _) \rightarrow \overline{\text{Hom}}(B, _) \rightarrow 0$$

Applying the contravariant exact functor D_A yields an exact sequence of functors in $(\text{mod}(\Lambda), \text{Ab})$

$$0 \rightarrow D_A \overline{\text{Hom}}(B, _) \rightarrow _ \otimes B \rightarrow _ \otimes I$$

which establishes that $D_A \overline{\text{Hom}}(B, _) \simeq _ \otimes B$. \square

Corollary 4.5. *The Auslander-Gruson-Jensen functor sends the cotorsion functor on left (right) modules to the injective torsion functor on right (left) modules. In short,*

$$D_A(\mathfrak{q}) \simeq \mathfrak{s}$$

Equivalently,

$$D_A(\text{Tr}(\mathcal{J}, _)^{-1}) \simeq \overrightarrow{\text{rej}}(_, \mathcal{F})$$

Proof. For the first isomorphism, set $B := \Lambda$. The second claim follows from Proposition 2.23 and (3.5) \square

Corollary 4.6. *Λ is right (respectively, left) absolutely pure if and only if every pure injective left (respectively, right) Λ -module is cotorsion-free.*

Proof. Proposition 2.20 shows that Λ is left absolutely pure if and only if \mathfrak{s} is the zero functor on right modules. By the preceding corollary, this is equivalent to $D_A(\mathfrak{q}) = 0$. By [4, Proposition 19], $D_A(\mathfrak{q}) = 0$ if and only if \mathfrak{q} vanishes on all pure injective left Λ -modules. \square

Remark 4.7. As we mentioned in Remark 3.15, Enochs and Jenda show that any pure injective is cotorsion in their sense. From the point of view of our definitions, their term “cotorsion” should be replaced by our term “cotorsion-free”. The just proved corollary shows what the correct statement should then be.

If M is a pure injective left Λ -module, then $_ \otimes M$ is injective in the functor category $(\text{mod}(\Lambda), \text{Ab})$. The functor D_L , which is the left adjoint to D_A , sends any injective $_ \otimes M$ to the representable $\text{Hom}(M, _)$. Because D_L is also right-exact, one can easily show the following.

Proposition 4.8. *For any pure injective left Λ -module M ,*

$$\overline{\text{Hom}}(M, -) \simeq D_L(- \otimes^{\overline{}} M)$$

Proof. For the pure injective module M take any monomorphism $0 \rightarrow M \rightarrow I$ with I injective. By applying the right-exact contravariant functor D_L to the exact sequence

$$0 \rightarrow - \otimes^{\overline{}} M \rightarrow - \otimes M \rightarrow - \otimes I$$

we have the exact sequence of functors

$$(I, -) \rightarrow (M, -) \rightarrow D_L(- \otimes^{\overline{}} M) \rightarrow 0$$

It follows that $D_L(- \otimes^{\overline{}} M) \simeq \overline{\text{Hom}}(M, -)$. \square

Corollary 4.9. *If ${}_{\Lambda}\Lambda$ is pure injective, then $\mathfrak{q} \simeq D_L(\mathfrak{s})$.* \square

Theorem 4.10. *Suppose that Λ is any ring for which there exists a finitely presented injective left module I and a monomorphism $0 \rightarrow {}_{\Lambda}\Lambda \rightarrow I$. Then the notions of torsion and cotorsion are dual. More precisely, the right adjoint*

$$D_R : (\text{mod}(\Lambda), \text{Ab}) \rightarrow \text{fp}(\text{Mod}(\Lambda^{op}), \text{Ab})$$

of D_A carries the injective torsion functor on right modules to the cotorsion functor on left modules, i.e.,

$$D_R(\mathfrak{s}) \simeq \mathfrak{q}$$

Proof. We have an exact sequence

$$0 \rightarrow \mathfrak{s} \rightarrow - \otimes \Lambda \rightarrow - \otimes I$$

Since this sequence lies entirely in $\text{fp}(\text{mod}(\Lambda), \text{Ab})$ and D_R is exact on this Serre subcategory, we have an exact sequence

$$D_R(- \otimes I) \rightarrow D_R(- \otimes \Lambda) \rightarrow D_R(\mathfrak{s}) \rightarrow 0$$

Because I, Λ are finitely presented modules, this is equivalent to the exact sequence

$$(I, -) \rightarrow (\Lambda, -) \rightarrow D_R(\mathfrak{s}) \rightarrow 0$$

Hence

$$D_R(\mathfrak{s}) \simeq \overline{\text{Hom}}(\Lambda, -) = \mathfrak{q}$$

\square

Corollary 4.11. *Let Λ be an artin algebra. Then $D_R(\mathfrak{s}) \simeq \mathfrak{q}$, where \mathfrak{s} and \mathfrak{q} have arbitrary opposite chiralities.* \square

In the case when Λ is an algebra over a commutative ring R , the connections between torsion and cotorsion can be made more pointed by utilizing the generalized Auslander-Reiten formula [12, Proposition 9.10]. Let \mathbf{J} be an injective R -module and $D_{\mathbf{J}} := \text{Hom}_R(-, \mathbf{J})$. By [ibid.], we have

$$D_{\mathbf{J}}(A \otimes^{\overline{}} B) \simeq \overline{\text{Hom}}(B, D_{\mathbf{J}}(A)),$$

where A is an arbitrary right Λ -module and B is an arbitrary left Λ -module. Specializing to the case $B = {}_{\Lambda}\Lambda$, we immediately have

Proposition 4.12. *In the above notation,*

$$D_{\mathbf{J}} \circ \mathfrak{s} \simeq \mathfrak{q} \circ D_{\mathbf{J}}$$

i.e., for each injective R -module \mathbf{J} and each right Λ -module A , we have an isomorphism $D_{\mathbf{J}}(\mathfrak{s}(A)) \simeq \mathfrak{q}(D_{\mathbf{J}}(A))$ which is functorial in A . \square

Corollary 4.13. *Let Λ be an arbitrary ring, $R := \mathbb{Z}$, and, for any right Λ -module A , let $A^+ := \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ be the character module of A . Then*

$$\mathfrak{s}(A)^+ \simeq \mathfrak{q}(A^+)$$

\square

Remark 4.14. In general, \mathfrak{q}^{-1} is not a radical or, equivalently, \mathfrak{q} is not idempotent, as can be seen from the following counterexample. Let Λ be a commutative local finite-dimensional \mathbb{k} -algebra over a field \mathbb{k} . Then $D_{\mathbb{k}}$ is a duality on the category of finite-dimensional Λ -modules. Applying $D_{\mathbb{k}}$ to an example from Remark 2.6, we have the desired counterexample. Details are left to the reader.

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